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Quantum Description of Anyons: Role of Contact Terms

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Abstract

We make an all-order analysis to establish the precise correspondence between nonrelativistic Chern-Simons quantum field theory and an appropriate first-quantized description. Physical role of the field-theoretic contact term in the context of renormalized perturbation theory is clarifed through their connection to self-adjoint extension of the Hamiltonian in the first-quantized approach. Our analysis provides a firm theoretical foundation on quantum field theories of nonrelativistic anyons.

I. INTRODUCTION

It is generally understood that for every Schrödinger (i.e., nonrelativistic) quantum field theory, there exists a corresponding quantum mechanical(or first-quantized) description [1]. This will certainly be the case with non-singular interactions, but becomes quite uncertain if singular interactions, such as the two-body δ-function potential, are involved. In two or more spatial dimensions, the δ-function potentials lead to nontrivial (i.e., interacting) systems only when infinite renormalization or self-adjoint extension of the Hamiltonian is taken into consideration [2–4]. Using the language of Schrödinger field theory, on the other hand, the δ-function potential between particles is formally represented by a local or contact interaction of the form $(\phi^\dagger(x)\phi(x))^2$. The Dyson-Feynman perturbation theory is then beset with ultraviolet divergences —to make sense out of this field theory, one must regularize and renormalize the amplitudes. Because of these complications it is not so easy to make a direct comparison between the first- and second- quantized approaches even for this relatively simple system. But, Bergman [5] demonstrated recently that the two approaches are in fact completely equivalent once the renormalized strength of the $(\phi^\dagger(x)\phi(x))^2$ interaction is chosen to be related in a specific way to the self-adjoint extension parameter entering the quantum mechanical approach.

In two spatial dimensions, we have another kind of system where similar problems arise naturally—those involving anyons [6]. Quantum mechanically, anyons can be regarded as flux-charge composites and so the relative dynamics of the two anyon system is essentially the Aharonov-Bohm scattering problem [7]. The latter problem has certain ambiguity as regards the choice of boundary condition at the singular point of the Aharonov-Bohm potential, namely, at the location of the flux line. According to the theory of self-adjoint extension, it is known that there exist a one-parameter family of acceptable boundary conditions [3,8], including as a special case the often-assumed hard-core boundary condition(appropriate to an impenetrable flux line) [7]. In perturbative treatments, however, this boundary condition is not easily incorporated and without paying due attention to it one ends up with a divergent

perturbation series [9]. Among various proposals made to amend this situation [10], a particularly interesting one is to introduce an extra δ-function potential of suitable strength [11], the role of which is to secure a finite(and correct) perturbation theory. The second-quantized description has certain ambiguity also. Naively, the field theory for anyons can be obtained if the Schrödinger field is coupled to a Chern-Simons gauge field [12,13]. But, once one takes renormalization into account, one is forced to allow the $(\phi^\dagger(x)\phi(x))^2$-type local interaction also in the theory [14]. Needless to say, it should be important to understand the physical role of such contact interaction term and also its significance within the first-quantized approach. This is especially so in view of possible relevance of the Chern-Simons field theory in some of most remarkable phenomena in planar physics such as the quantum Hall effect.

A serious study on the above issue was made recently by Amelino-Camelia and Bak [15]. They showed that lower-order field theory calculations of the scattering amplitudes agree with the corresponding quantum mechanical results obtained using the method of self-adjoint extension, provided the strength of the $(\phi^\dagger(x)\phi(x))^2$-interaction is suitably related to the self-adjoint extension parameter. Note that this situation is entirely analogous to the pure δ-function case mentioned earlier. A new feature is that, if the strength of the $(\phi^\dagger(x)\phi(x))^2$-interaction is equal to the critical value appropriate to the so-called self-dual limit [13], the field theory turns out to be ultraviolet finite [14] and yields the anyon scattering amplitude consistent with the scale-invariant boundary conditions. This last fact has now been confirmed by one of us to *all* orders in perturbation theory [16]. With these developments it should be useful to have a self-contained account on quantum description of anyons which takes contact terms properly into consideration, in both first- and second-quantized formulations. This will, above all, serve to show the essential equivalence of the two formulations and illuminate the physical significance of the contact term entering either formulation. The present work has been written precisely for this purpose.

This paper is organized as follows. In Sec. II we reconsider quantum mechanical description of anyons and explain in particular how the boundary condition, needed at the

coincidence point of two anyons, can be implemented through the introduction of appropriate contact terms in the Hamiltonian. Note that we here consider not just the hard-core boundary condition but the general boundary condition allowed by the self-adjoint extension method [The corresponding anyons were called 'colliding anyons' in Ref. [15]]. Sec. III is concerned with the Chern-Simons field theory description of anyons, with the $(\phi^\dagger(x)\phi(x))^2$-type contact terms included for the sake of renormalizability. In this field theory context we elaborate on the all-order analysis of the two-anyon s-wave scattering amplitude, a brief account of which was given for the first time in Ref. [16]. Based on this analysis, we then clarify the connection between the field theoretic formulation and the quantum mechanical description. Sec. IV contains the summary and discussions.

II. FIRST-QUANTIZED DESCRIPTIONS OF ANYONS WITH BOUNDARY-CONDITION IMPLEMENTING CONTACT TERMS

Anyons, which are realized only in planar physics, can be described using either bosonic or fermionic description. Taking bosonic description, one may specify quantum dynamics of a system of anyons by the particle Hamiltonian

$$H = \sum_n \frac{1}{2m}(\mathbf{p}_n - \alpha \mathbf{A}_n)^2 + \frac{1}{2}\sum_{n,m(\neq n)} U(|\mathbf{r}_n - \mathbf{r}_m|), \tag{2.1}$$

$$A_n^i \equiv \epsilon^{ij} \sum_{m(\neq n)} \frac{x_n^j - x_m^j}{|\mathbf{r}_n - \mathbf{r}_m|^2}, \tag{2.2}$$

where $\mathbf{r}_n \equiv (x_n^1, x_n^2)$ denotes the position of the n-th particle, and $U(|\mathbf{r}_n - \mathbf{r}_m|)$ a non-singular two-body potential included for generality. The vector potential $\mathbf{A}_n \equiv (A_n^1, A_n^2)$ seen by particle n is that of point vortices carried by all the other particles, and it is the resulting Aharonov-Bohm-type interference effect that is responsible for the anyonic behavior of particles. The parameter α, called 'statistical parameter', characterizes the type of anyons (i.e., their statistics) and without loss of generality α may be restricted to the interval $(-1,1)$.

But, due to the singular nature of the vector potential (2.2) at points $\mathbf{r}_n = \mathbf{r}_m$, the information given above does *not* specify the system completely — a suitable boundary condition

at locations of singularity must be posited. There exist a class of boundary conditions (see below) all of which are in fact realizable with the help of suitable regularization procedures. Thus, anyons are further classified by the nature of the boundary condition chosen at two-anyon intersection points. Also this extra specification is something that *has to be made* and should be taken into account, say, in all approximate treatments. Divergences encountered in naive perturbation theory (with $|\alpha| \ll 1$) for the anyon system can be ascribed to the ill-defined nature of the problem caused by not fixing the boundary condition [10]. Under the circumstance that the boundary condition cannot be ignored, it will then be natural to ask whether the boundary condition in question may be implemented by having instead an appropriate contact interaction term in the Hamiltonian. That is possible, as we will show below. This entirely-Hamiltonian description for anyons will allow a straightforward application of perturbation theory and also serve as a bridge to the field-theoretic description.

To study the boundary condition problem mentioned above, we may concentrate on the relative dynamics in the two anyon sector. It will be governed by the Hamiltonian

$$H_{rel} = \frac{1}{m}(\mathbf{p} - \alpha\mathbf{A})^2 + U(r), \tag{2.3}$$

$$A^i \equiv \frac{\epsilon^{ij} x^j}{r^2}, \qquad (r \equiv |\mathbf{r}|) \tag{2.4}$$

where $\mathbf{r} \equiv (x^1, x^2)$ and $\mathbf{p}$ denote relative position and momentum. Furthermore, since the boundary condition to be chosen at $r = 0$ has an effect on the s-wave only, it will suffice to consider the s-wave Hamiltonian. Throughout this paper, we also set $\hbar = 1$. The corresponding Schrödinger equation reads

$$\left[-\frac{1}{r}\frac{d}{dr} r \frac{d}{dr} + \frac{\alpha^2}{r^2} + mU(r) \right] \psi(r) = k^2 \psi(r), \qquad (r > 0) \tag{2.5}$$

which has the general solution of the form (for $\alpha \neq 0$)

$$\psi(r) = A\mathcal{J}_{|\alpha|}(kr) + B\mathcal{J}_{-|\alpha|}(kr), \tag{2.6}$$

where A, B are arbitrary constants. Here, $\mathcal{J}_{\pm|\alpha|}(kr)$ represent two linearly independent solutions of Eq. (2.5) with the following small-r behaviors

$$\mathcal{J}_{\pm|\alpha|}(kr) \sim \frac{1}{\Gamma(1 \pm |\alpha|)} \left(\frac{kr}{2}\right)^{\pm|\alpha|} [1 + \mathcal{O}(r)], \quad r \to 0. \tag{2.7}$$

Note that we have assumed a sufficient regularity of $U(r)$ at $r = 0$, and the functions $\mathcal{J}_{\pm|\alpha|}(kr)$ have been normalized in such a way that they reduce to ordinary Bessel functions, i.e., $J_{\pm|\alpha|}(kr)$ if $U(r)$ is taken to vanish. The wave function in Eq. (2.6) is not regular at $r = 0$, but still square integrable (with $|\alpha| < 1$) for arbitrary finite values of A and B. This is in marked contrast with the case of higher partial waves where only one type of solution is square integrable. In the present case we may pick an appropriate boundary condition at the origin in accordance with the method of self-adjoint extension of the Hamiltonian. This leads to a one-parameter family of boundary condition [3,8]

$$\lim_{r\to 0} \left[r^{|\alpha|}\psi_\theta(\mathbf{r}) - \frac{\tan\theta}{\mu^{2|\alpha|}} \frac{\Gamma(1+|\alpha|)}{\Gamma(1-|\alpha|)} \frac{d}{d\,(r^{2|\alpha|})} \left(r^{|\alpha|}\psi_\theta(\mathbf{r})\right)\right] = 0 \tag{2.8}$$

with the corresponding solution given as

$$\psi_\theta(r) = (const.) \left[\mathcal{J}_{|\alpha|}(kr) + \tan\theta \left(\frac{k}{\mu}\right)^{2|\alpha|} \mathcal{J}_{-|\alpha|}(kr) \right], \tag{2.9}$$

where θ is a dimensionless real parameter and μ a reference scale. The hard-core boundary condition (i.e., $\psi(0) = 0$), which is often assumed in the literature, corresponds to the choice $\theta = 0$ but, clearly, there is no a priori reason to favor this particular case.

We now discuss how we can implement the boundary condition (2.8) *dynamically*, *viz.*, by introducing a contact interaction term of the form

$$H_c = \lambda_\theta(r)\delta^2(\mathbf{r}) \tag{2.10}$$

in the Hamiltonian. [In the notation of Eq. (2.1), this contact term translates into two-body interaction of the form $H_c = \frac{1}{2}\sum_{n,m(\neq n)} \lambda_\theta(|\mathbf{r}_n - \mathbf{r}_m|)\delta^2(\mathbf{r}_n - \mathbf{r}_m)$.] In Eq. (2.10) we have written a δ-function multiplied by an r-dependent function, an apparently redundant expression. But, as we shall see soon, it has a reason.[1] If we form the new relative Hamiltonian

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[1]This is related to the fact that strength of the δ-function in our case needs renormalization(except for the special cases of $\theta = 0$ or $\frac{\pi}{2}$). The situation here is analogous to the case of a pure δ-function potential problem, as discussed in Refs. [2–4].

$$\tilde{H}_{rel} = H_{rel} + H_c, \tag{2.11}$$

the function ψ_θ given in Eq. (2.9) will then have to be its eigenstate(when all ill-behaved quantities are interpreted in a suitably regularized sense). This demands especially that

$$\langle\psi|\tilde{H}_{rel}|\psi_\theta\rangle = \frac{k^2}{m}\langle\psi|\psi_\theta\rangle \tag{2.12}$$

or

$$\int d^2\mathbf{r}\psi^*(r)\left\{\frac{1}{m}(-i\nabla - \alpha\mathbf{A})^2 + U(r) + \lambda_\theta(r)\delta^2(\mathbf{r}) - \frac{k^2}{m}\right\}\psi_\theta(r) = 0, \tag{2.13}$$

where $\psi(r)$ can be an arbitrary function of the form (2.6). One must choose $\lambda_\theta(r)$ such that Eq. (2.13) may hold. In using the condition (2.13) it is to be noted that, since both $\psi^\dagger(r)$ and $\psi_\theta(r)$ are in general not regular at $r = 0$, the operation with the δ-function term demands much care and especially one may not insert $\lambda_\theta(0)$ for $\lambda_\theta(r)$. Actually, there are other ill-behaved contributions in Eq. (2.13) also. See below.

To give a precise meaning to Eq. (2.13) and also to the Schrödinger equation with $\tilde{H}_{rel}$ (in the range $r \geq 0$), we must introduce a suitable regularization procedure as regards the singularity at $r = 0$. This may be effected by replacing the vector potential (2.4) by the regularized expression

$$A^{i(\epsilon)} = \frac{1}{r+\epsilon}\left(\frac{\epsilon^{ij}x^j}{r}\right) \tag{2.14}$$

and the δ-function $\delta^2(\mathbf{r})$ by the regularized one

$$\delta^2(\mathbf{r};\epsilon) = \frac{1}{2\pi}\nabla^2\log(r+\epsilon) = \frac{1}{2\pi r}\frac{\epsilon}{(r+\epsilon)^2}, \tag{2.15}$$

with $\epsilon \to 0+$ understood if there is no further dangerous manipulation left. Our regularization procedure is simply to replace r by $r + \epsilon$, while leaving all angular dependences intact. Then it is possible to show that the functions obtained from $\mathcal{J}_{\pm|\alpha|}(kr)$ by the simple substitution $r \to r + \epsilon$ satisfy the equations (for $r \geq 0$)

$$\left[\frac{1}{m}\left(-i\nabla - \alpha\mathbf{A}^{(\epsilon)}\right)^2 + U(r) - \frac{k^2}{m}\right]\mathcal{J}_{\pm|\alpha|}(k(r+\epsilon))$$

$$
\begin{aligned}
&= \left[\frac{1}{m}\left(-\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}+\frac{\alpha^2}{(r+\epsilon)^2}\right)+U(r)-\frac{k^2}{m}\right]\mathcal{J}_{\pm|\alpha|}(k(r+\epsilon)) \\
&= \mp\frac{|\alpha|}{m}\frac{\left[\frac{k(r+\epsilon)}{2}\right]^{\pm|\alpha|}}{\Gamma(1\pm|\alpha|)}\frac{\epsilon}{r(r+\epsilon)^2} \\
&\equiv \mp\frac{2\pi|\alpha|}{m}\frac{\left[\frac{k(r+\epsilon)}{2}\right]^{\pm|\alpha|}}{\Gamma(1\pm|\alpha|)}\delta^2(\mathbf{r};\epsilon).
\end{aligned} \tag{2.16}
$$

These follow readily from the behaviors (2.7) and the fact that $\mathcal{J}_{\pm|\alpha|}(kr)$ solve the differential equation (2.5). Using Eq. (2.16) we may thus conclude that the function $\psi_\theta(r+\epsilon)$, given by the expression (2.9) with the substitution $r \to r+\epsilon$, satisfies the relation

$$
\begin{aligned}
&\left[\frac{1}{m}\left(-i\nabla-\alpha\mathbf{A}^{(\epsilon)}\right)^2+U(r)-\frac{k^2}{m}\right]\psi_\theta(r+\epsilon) \\
&= -\frac{2\pi|\alpha|}{m}\left\{\frac{\frac{1}{\Gamma(1+|\alpha|)}\left[\frac{\mu(r+\epsilon)}{2}\right]^{|\alpha|}-\tan\theta\frac{1}{\Gamma(1-|\alpha|)}\left[\frac{\mu(r+\epsilon)}{2}\right]^{-|\alpha|}}{\frac{1}{\Gamma(1+|\alpha|)}\left[\frac{\mu(r+\epsilon)}{2}\right]^{|\alpha|}+\tan\theta\frac{1}{\Gamma(1-|\alpha|)}\left[\frac{\mu(r+\epsilon)}{2}\right]^{-|\alpha|}}\right\}\delta^2(\mathbf{r};\epsilon)\psi_\theta(r+\epsilon).
\end{aligned} \tag{2.17}
$$

Note that what we have on the right hand side of this equation can yield a nontrivial contribution, say, to the matrix element formed with the general state given in Eq. (2.6).

From Eq. (2.17) the precise form of the necessary contact interaction term can be inferred: the quantity multiplying $\psi(r+\epsilon)$ in the right hand side of Eq. (2.17) should be identified with $-\lambda_\theta(r)\delta^2(\mathbf{r})$ or, in a regularized form, with $-\lambda_\theta(r)\delta^2(\mathbf{r};\epsilon)$. That is[2],

$$
\begin{aligned}
\lambda_\theta(r) &= -\frac{2\pi|\alpha|}{m}\left\{\frac{\Gamma(1+|\alpha|)\tan\theta-\Gamma(1-|\alpha|)\left[\frac{\mu(r+\epsilon)}{2}\right]^{2|\alpha|}}{\Gamma(1+|\alpha|)\tan\theta+\Gamma(1-|\alpha|)\left[\frac{\mu(r+\epsilon)}{2}\right]^{2|\alpha|}}\right\} \\
&= \left[\begin{array}{ll} \dfrac{2\pi|\alpha|}{m}, & \theta=0 \\ -\dfrac{2\pi|\alpha|}{m}\left\{1-2\dfrac{\Gamma(1-|\alpha|)}{\Gamma(1+|\alpha|)}\cot\theta\left[\dfrac{\mu(r+\epsilon)}{2}\right]^{2|\alpha|}\right\}, & \theta\neq 0. \end{array}\right.
\end{aligned} \tag{2.18}
$$

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[2]If one wishes, the form given in Eq. (2.18) for λ_θ may be replaced by another expression involving only the regularization parameter ϵ but not r. [One may use the integral condition (2.13) for this purpose.] But, with a regularized δ-function brought in, there is no reason to favor such expression in particular; indeed, our development leads quite naturally to the form (2.18).

With the thus-constructed contact term included in $\tilde{H}_{rel}$, this new Hamiltonian, without any separate consideration of the boundary condition at the origin, will select the function-type $\psi(r) = \psi_\theta(r)$(i.e., the $\epsilon \to 0+$ limit of $\psi_\theta(r+\epsilon)$) as its only acceptable eigenfunction type. Especially, the hard-core boundary condition (i.e., the $\theta = 0$ case) is implemented by a particularly simple contact Hamiltonian, $H_c = \frac{2\pi|\alpha|}{m}\delta^2(\mathbf{r})$. This repulsive contact term is precisely what has been suggested by the authors of Ref. [11] as the extra interaction needed to cancel perturbation-theory divergences. Also interesting is the fact that, since

$$\begin{aligned} B &\equiv \epsilon_{ij}\partial_i A_j^{(\epsilon)} \\ &= -\frac{\epsilon}{r(r+\epsilon)^2} = -2\pi\delta^2(\mathbf{r};\epsilon), \end{aligned} \tag{2.19}$$

the net regularized relative Hamiltonian $\tilde{H}_{rel}$ (see Eq. (2.11) in the $\theta = 0$ case assumes the form of the (spin-fixed) two-dimensional Pauli Hamiltonian withthe vector potential $\mathbf{A}^{(\epsilon)}$ and the scalar potential $U(r)$.

Note that, even for the same given boundary condition, the contact Hamiltonian may have a slightly different look if the method of regularization is different. For example, suppose we replace the vector potential (2.4) by that of finite-radius flux tube of radius r_0 (with the magnetic field confined to the surface of the tube) [17], i.e. by

$$A^{i(r_0)} = \left[\begin{array}{ll} 0, & r < r_0 \\ \dfrac{\epsilon^{ij}x^j}{r^2}, & r > r_0 \end{array} \right. \tag{2.20}$$

Then, for the regularized form of the contact Hamiltonian, the ring potential

$$H_c = \frac{\bar{\lambda}_\theta}{2\pi r_0}\delta(r - r_0) \qquad \Big(\approx \bar{\lambda}_\theta \delta^2(r)\Big) \tag{2.21}$$

will especially be appropriate. Here the strength $\bar{\lambda}_\theta$ is to be chosen such that the boundary condition (2.8) may be realized for $r_0 \to 0$. Using this form, Hagen [18] demonstrated how one may implement the hard-core boundary condition. This can easily be generalized to accomodate more general boundary condition in Eq. (2.8). Now the eigenfunction of H_{rel} will have the form (up to an overall multiplicative constant)

$$\psi(r) = \begin{bmatrix} \mathcal{J}_0(kr), & r < r_0 \\ A\mathcal{J}_{|\alpha|}(kr) + B\mathcal{J}_{-|\alpha|}(kr), & r > r_0 \end{bmatrix} \tag{2.22}$$

with the constants A and B determined by the conditions

$$\begin{aligned} &\psi(r_0+\epsilon) - \psi(r_0-\epsilon) = 0, \\ &\left.\frac{d\psi}{dr}\right|_{r=r_0+\epsilon} - \left.\frac{d\psi}{dr}\right|_{r=r_0-\epsilon} = m\frac{\bar{\lambda}_\theta}{2\pi r_0}\psi(r_0). \end{aligned} \tag{2.23}$$

For sufficiently small kr_0, Eq. (2.23) imply

$$\begin{aligned} A\frac{1}{\Gamma(1+|\alpha|)}\left(\frac{kr_0}{2}\right)^{|\alpha|} + B\frac{1}{\Gamma(1-|\alpha|)}\left(\frac{kr_0}{2}\right)^{-|\alpha|} &= 1, \\ A\frac{|\alpha|}{\Gamma(1+|\alpha|)}\left(\frac{kr_0}{2}\right)^{|\alpha|} - B\frac{|\alpha|}{\Gamma(1-|\alpha|)}\left(\frac{kr_0}{2}\right)^{-|\alpha|} &= \frac{m}{2\pi}\bar{\lambda}_\theta, \end{aligned} \tag{2.24}$$

and hence we obtain the ratio

$$\frac{B}{A} = \frac{\Gamma(1-|\alpha|)\left(|\alpha| - \frac{m\bar{\lambda}_\theta}{2\pi}\right)}{\Gamma(1+|\alpha|)\left(|\alpha| + \frac{m\bar{\lambda}_\theta}{2\pi}\right)}\left(\frac{kr_0}{2}\right)^{2|\alpha|}. \tag{2.25}$$

On the other hand, if we compare the above wave function in the region $r > r_0$ with the form given in Eq. (2.9), we are led to set

$$\frac{B}{A} = \tan\theta\left(\frac{k}{\mu}\right)^{2|\alpha|}. \tag{2.26}$$

From Eq. (2.25) and Eq. (2.26) we thus see that the strength $\bar{\lambda}_\theta$ of the given contact term should be chosen as

$$\bar{\lambda}_\theta = -\frac{2\pi|\alpha|}{m}\frac{\Gamma(1+|\alpha|)\tan\theta - \Gamma(1-|\alpha|)\left(\frac{\mu r_0}{2}\right)^{2|\alpha|}}{\Gamma(1+|\alpha|)\tan\theta + \Gamma(1-|\alpha|)\left(\frac{\mu r_0}{2}\right)^{2|\alpha|}}. \tag{2.27}$$

Note that, for $\theta = 0$, we again find the value $\bar{\lambda}_{\theta=0} = \frac{2\pi|\alpha|}{m}$.

We have so far shown that an anyon system can be specified solely by the Hamiltonian, only when one takes into account a suitable contact interaction term. It is needed to implement the boundary condition chosen at the two-particle intersection point. For the special case of $\theta = 0$ or $\frac{\pi}{2}$, this contact Hamiltonian assumes a particularly simple form, *viz.*, $H_c = \frac{2\pi|\alpha|}{m}\delta^2(\mathbf{r})$ for $\theta = 0$ and $H_c = -\frac{2\pi|\alpha|}{m}\delta^2(\mathbf{r})$ for $\theta = \frac{\pi}{2}$. In fact, only for $\theta = 0$ or $\theta = \frac{\pi}{2}$,

H_c and also $\psi_\theta(r)$ in Eq. (2.9) (up to an irrelevant overall constant) become independent of our reference scale μ; this is an evidence of the *scale invariance* in the system. Here one might suspect that, for a general N-anyon Hamiltonian, contact interaction terms involving more than two particles may have to be introduced as well. We strongly believe that these should be unnecessary, i.e., two-body contact interactions we have discussed suffice. This is supported by the perturbative analysis of an N-anyon system (in Ref. [11], for example) and also by the renormalization counterterm structure in the field theoretic approach.

Before closing this section, we will give the explicit expression for the s-wave scattering amplitude of two anyons when the two-body potential $U(\mathbf{r}_n - \mathbf{r}_m)$ is taken to be zero. Following the analysis given in Ref. [7] yields the amplitude

$$A_s(p) = (e^{-i\pi|\alpha|} - 1)\frac{(\mu/p)^{2|\alpha|} - \tan\theta}{(\mu/p)^{2|\alpha|} + e^{-i\pi|\alpha|}\tan\theta}, \tag{2.28}$$

where p is the magnitude of the relative momentum. Defining

$$\lambda_{ren} \equiv -\frac{4\pi|\alpha|}{m}\frac{\tan\theta - 1}{\tan\theta + 1}, \tag{2.29}$$

this may be rewritten as

$$\begin{aligned} A_s(p) &= e^{-i\pi|\alpha|} - 1 - (e^{i\pi|\alpha|} - e^{-i\pi|\alpha|})\frac{\lambda_{ren} - \lambda_0}{\lambda_0 - \lambda_{ren} + (\lambda_0 + \lambda_{ren})(\mu/p)^{2|\alpha|}e^{i\pi|\alpha|}} \\ &= e^{-i\pi|\alpha|} - 1 - \frac{e^{i\pi|\alpha|} - e^{-i\pi|\alpha|}}{2\lambda_0}(\lambda_{ren} - \lambda_0) \\ &\quad + \frac{e^{i\pi|\alpha|} - e^{-i\pi|\alpha|}}{2\lambda_0}\frac{\frac{(\mu/p)^{2|\alpha|}e^{i\pi|\alpha|} - 1}{2\lambda_0}}{1 + (\lambda_{ren} + \lambda_0)\frac{(\mu/p)^{2|\alpha|}e^{i\pi|\alpha|} - 1}{2\lambda_0}}(\lambda_{ren}^2 - \lambda_0^2), \end{aligned} \tag{2.30}$$

where $\lambda_0 = \frac{2\pi|\alpha|}{m}$. [Note that the appropriate expression with the hard-core boundary condition, as adopted in Ref. [7], is obtained if we set $\lambda_{ren} = \lambda_0$ in Eq. (2.30).] We may now of course look on this result as that corresponding to the system defined by the Hamiltonian (2.11) (with $U \equiv 0$) which contains the contact interaction. We will make use of this result in the next section.

III. QUANTUM FIELD THEORETIC DESCRIPTION OF ANYONS

When interactions involved are nonsingular, a Schrödinger quantum field theory is known to be completely equivalent to nonrelativistic quantum mechanics of many particles [1]. Needless to say, Feynman diagram approach in many body theory is an important byproduct of this correspondence. But, in the presence of local or contact interactions, the singular nature of interaction makes the situation no longer simple—both infinite renormalization (in the field theoretic approach) and self-adjoint extension of the Hamiltonian (in the quantum mechanical approach) take parts in any discussion purporting to establish the analogous correspondence. In the latter case, we are not aware of any general argument as regards the nature of such correspondence and so each model system has to discussed separately. [The main obstacle to giving *general* arguments stems from the big difference in the flavored language for the two approaches, one diagrammatical (and in momentum space) and the other in the form of differential equations (in position space)]. The anyon system involves singular interactions and this issue arises naturally. We will below give a field theory description of anyons (including renormalization effects) and then relate it to the quantum mechanical description of the previous section. For earlier related works, see Refs. [5], [14] and [15].

We begin by specifying our candidate quantum field theory for anyons. It is a (2+1)-D nonrelativistic system described by the Lagrangian density

$$\mathcal{L} = \frac{\kappa}{2}\partial_t \mathbf{A} \times \mathbf{A} - \kappa A_0 B + \phi^\dagger (iD_t + \frac{\mathbf{D}^2}{2m})\phi - \frac{\lambda}{2}\phi^\dagger\phi^\dagger\phi\phi, \tag{3.1}$$

where ϕ is a bosonic field, $\mathbf{A}$=(A_1,A_2) denotes a Chern-Simons gauge field, B=$\epsilon_{ij}\partial_i A_j \equiv \nabla \times \mathbf{A}$ and the covariant derivatives are

$$\begin{aligned} D_t &= \partial_t + ieA_0 \\ \mathbf{D} &= \nabla - ie\mathbf{A}. \end{aligned} \tag{3.2}$$

Without the last term in Eq. (3.1), this model was first considered by Hagen [12], But the last contact interaction term, first considered in Ref. [13], is necessary to ensure the

renormalizability of the theory. As it turns out, this additional term is of crucial importance in the field theoretic treatment of anyons. For a comprehensive review on various aspects concerning the above theory, readers may consult Ref. [19].

Setting aside the renormalization problem for the moment, it might be useful to reproduce quantum mechanical description corresponding to the above theory by the standard many-body-theory procedure. First, using the 'Gauss' constraint

$$\nabla \times \mathbf{A} = -\frac{e}{\kappa}\phi^\dagger\phi, \tag{3.3}$$

the gauge fields $\mathbf{A}$ may be expressed in terms of the matter fields (in the Coulomb gauge) as

$$\mathbf{A}(\mathbf{r},t) = -\frac{e}{\kappa}\nabla \times \int d^2r' G(\mathbf{r}-\mathbf{r}')\phi^\dagger(\mathbf{r}',t)\phi(\mathbf{r}',t), \tag{3.4}$$

where $G(\mathbf{r})$ is the Green's function of the two-dimensional Laplacian

$$G(\mathbf{r}) = \frac{1}{2\pi}\ln|\mathbf{r}|. \tag{3.5}$$

Assuming Eq. (3.4), the Hamiltonian can now be identified with

$$H = \int d^2r \left[\frac{1}{2m}(\mathbf{D}\phi)^\dagger \cdot (\mathbf{D}\phi) + \frac{\lambda}{2}\phi^\dagger\phi^\dagger\phi\phi\right], \tag{3.6}$$

Then, defining the N-particle Schrödinger wave function

$$\Phi(\mathbf{r_1},\cdots,\mathbf{r_N},t) \equiv \frac{1}{N!}\langle 0|\phi(\mathbf{r}_1,t)\cdots\phi(\mathbf{r}_N,t)|\Phi\rangle \tag{3.7}$$

and using the canonical commutation relations

$$[\phi(\mathbf{r},t),\phi(\mathbf{r}',t')] = [\phi^\dagger(\mathbf{r},t),\phi^\dagger(\mathbf{r}',t')] = 0, \quad [\phi(\mathbf{r},t),\phi^\dagger(\mathbf{r}',t')] = \delta(\mathbf{r}-\mathbf{r}'), \tag{3.8}$$

it is straightforward (but tedious) to derive the Schrödinger equation of the form [13,20]

$$\begin{aligned} i\frac{\partial}{\partial t}\Phi(\mathbf{r_1},\cdots,\mathbf{r_N},t) = \Bigg\{ &-\frac{1}{2m}\sum_n \left[\nabla_n - \frac{ie^2}{\kappa}\nabla_n \times \left(\sum_{m(\neq n)} G(\mathbf{r}_n-\mathbf{r}_m)\right)\right]^2 \\ &+ \frac{\lambda}{2}\sum_{n,m(\neq n)} \delta(\mathbf{r}_n-\mathbf{r}_m)\Bigg\}\Phi(\mathbf{r_1},\cdots,\mathbf{r_N},t). \end{aligned} \tag{3.9}$$

What we have in Eq. (3.9) has the appearance of the Schrödinger equation for the anyon system, with $\alpha = \frac{e^2}{2\pi\kappa}$ and $U(|\mathbf{r}_n - \mathbf{r}_m|) = 0$ in the notation of Sec. II; the δ-function potential in Eq. (3.9), which originates from the $\phi^\dagger\phi^\dagger\phi\phi$ coupling in Eq. (3.1), may be viewed as the boundary-condition implementing term at two-particle intersection points. But the above argument suggests at most the formal correspondence for *bare* amplitudes (with singular interactions replaced by suitably regularized one, as we have done in Sec. II). Our goal is to find the correspondence between well-defined renormalized amplitudes of the two approaches. In the quantum mechanical description, we have invoked the method of self-adjoint extension to find such well-defined two-particle scattering amplitude which depends on the self-adjoint extension parameter θ (or on λ_{ren}, as defined in Eq. (2.29)) but not on the bare contact coupling λ. To be able to make an unambiguous comparison, the corresponding renormalized amplitude in the field theory context will be obtained below. Here it is perhaps worthwhile to remark that the correspondence we found above (for bare quantities) may be more than a formal one if the theory is free from ultraviolet divergences; this happens for the λ-value equal to $\pm\frac{2\pi|\alpha|}{m}$, for which we have a scale-invariant system. In this connection, see the paragraph following Eq. (2.27) and the discussion following immediately after Eq. (3.39) below.

Given the Lagrangian density (3.1), Feynman rules are as follows. The nonrelativistic boson propagator in momentum space is

$$\Delta(k) = \frac{1}{\left(k_0 - \frac{\mathbf{k}^2}{2m} + i\epsilon\right)}, \tag{3.10}$$

so that we have[3]

$$-i\langle 0|T\phi(x)\phi^*(0)|0\rangle = \int \frac{d^3k}{(2\pi)^3}\Delta(k)e^{-i(k_0x_0 - \mathbf{k}\cdot\mathbf{x})}. \tag{3.11}$$

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[3]Here and also in Eq. (3.13), one usually has free-field operators ϕ^f and A^f instead of full fields ϕ and A. But, because of the reason to be explained shortly, there is no mass and field renormalization in our theory so that the free-field restriction can be omitted.

Introducing the gauge fixing term

$$\mathcal{L}_{gf} = -\frac{1}{\xi}(\nabla \cdot \mathbf{A})^2 \tag{3.12}$$

and then considering the limit $\xi \to 0$, the only nonvanishing components of the Chern-simons gauge field-propagator is easily found to be

$$-i\langle 0|TA_i(x)A_0(0)|0\rangle = \int \frac{d^3k}{(2\pi)^3} D_{i0}(k) e^{-i(k_0x_0 - \mathbf{k}\cdot\mathbf{x})}. \tag{3.13}$$

with

$$D_{i0}(k) = -D_{0i}(k) = \frac{i\epsilon^{ij}k_j}{\kappa \mathbf{k}^2}, \tag{3.14}$$

There are four interaction vertices as shown in Fig.1. Three of them coming from the covariant derivative terms, are given as

$$\Gamma_0 = -ie, \tag{3.15a}$$

$$\Gamma_i = \frac{ie}{2m}(p_i + p_i'), \tag{3.15b}$$

$$\Gamma_{ij} = -\frac{ie^2}{m}\delta_{ij}, \tag{3.15c}$$

while the other from the contact interaction term reads

$$\Gamma_\lambda = -2i\lambda. \tag{3.16}$$

With our gauge choice (that is, the Coulomb gauge) and normal ordering of contact interaction term $\frac{\lambda}{2}\phi^\dagger\phi^\dagger\phi\phi$, diagrams in Fig.2 give zero contribution. So there are no renormalization of mass, field and charge e in this nonrelativistic theory. It is only the strength of contact interaction which has a nontrivial renormalization effect. This requires a detailed study of two-particle scattering amplitude.

The two-particle scattering amplitude can be described in terms of the effective two-particle interaction [1] Γ, represented by (see Fig.3)

$$\begin{aligned}\Gamma(p,p';q,q') = K(p,p';q,q') + \int \frac{d^3k}{(2\pi)^3} K(p,p';q,q') i\Delta(k) i\Delta(p+p'-k) \\ \cdot K(k,p+p'-k;q,q') + \cdots,\end{aligned} \tag{3.17}$$

where K denotes the two-particle-irreducible kernel. The *entire* nonvanishing graphs which are not reducible by cutting two matter lines are those shown in Fig.4, and the full kernel K can be identified with the sum of K_a, K_b and K_c. [Note that, in this nonrelativistic field theory, graphs like those shown in Fig.5 vanish indentically.]

Using the Feynman rules, one then finds that the quantities K_a, K_b and K_c are given by the following expressions (which are independent of energy variables):

$$K_a(p,p';q,q') = \frac{e^2}{m\kappa}\frac{(\mathbf{q}-\mathbf{p})\times(\mathbf{p}-\mathbf{p}')}{(\mathbf{p}-\mathbf{q})^2} \tag{3.18a}$$

$$K_b(p,p';q,q') = -i\frac{e^4}{4\pi m\kappa^2}\ln\frac{\Lambda^2}{(\mathbf{p}-\mathbf{q})^2} \tag{3.18b}$$

$$K_c(p,p';q,q') = -i\lambda \tag{3.18c}$$

In Eq. (3.18b), Λ is an ultraviolet momentum cutoff and one can obtain the given expression as

$$\begin{aligned}
K_b(p,p';q,q') &= -i\frac{e^4}{2\pi m\kappa^2}\int\frac{d^3k}{(2\pi)^3}\frac{i}{p_0+k_0-\frac{(\mathbf{p}+\mathbf{k})^2}{2m}+i\epsilon}\,\frac{\mathbf{k}\cdot(\mathbf{q}-\mathbf{p}-\mathbf{k})}{\mathbf{k}^2(\mathbf{q}-\mathbf{p}-\mathbf{k})^2}\\
&\quad+(\mathbf{p}\to\mathbf{p}',\mathbf{q}\to\mathbf{q}')\\
&= -i\frac{e^4}{4\pi m\kappa^2}\int\frac{d^2\mathbf{k}}{(2\pi)^2}\frac{\mathbf{k}\cdot(\mathbf{q}-\mathbf{p}-\mathbf{k})}{\mathbf{k}^2(\mathbf{q}-\mathbf{p}-\mathbf{k})^2}+(\mathbf{p}\to\mathbf{p}',\mathbf{q}\to\mathbf{q}')\\
&= -i\frac{e^4}{4\pi m\kappa^2}\int_0^{2\pi}d\varphi\int_0^{\Lambda}\frac{|\mathbf{k}|d|\mathbf{k}|}{(2\pi)^2}\frac{|\mathbf{k}||\mathbf{q}-\mathbf{p}|\cos\varphi-\mathbf{k}^2}{\mathbf{k}^2(\mathbf{k}^2-2|\mathbf{k}||\mathbf{q}-\mathbf{p}|\cos\varphi+(\mathbf{q}-\mathbf{p})^2)}\\
&\quad+(\mathbf{p}\to\mathbf{p}',\mathbf{q}\to\mathbf{q}')\\
&= -i\frac{e^4}{4\pi m\kappa^2}\ln\frac{\Lambda^2}{(\mathbf{p}-\mathbf{q})^2}
\end{aligned} \tag{3.19}$$

Also note that the equation (3.17) for Γ may be recast as the integral equation

$$\begin{aligned}
\Gamma(p,p';q,q') &= K(p,p';q,q') + \int\frac{d^3k}{(2\pi)^3}K(p,p';k,p+p'-k)\frac{i}{k_0-\frac{\mathbf{k}^2}{2m}+i\epsilon}\\
&\quad\cdot\frac{i}{p_0+p_0'-k_0-\frac{(\mathbf{p}+\mathbf{p}'-\mathbf{k})^2}{2m}+i\epsilon}\Gamma(k,p+p'-k;q,q'),
\end{aligned} \tag{3.20}$$

which is the Bethe-Salpeter equation. From Eq. (3.20) we see that Γ depends not on p_0 or p_0' separately but on the sum p_0+p_0' only. This allows one to perform the k_0-intergration immediately (using Cauchy's theorem), to yield the equation

$$\Gamma(p,p';q,q') = K(p,p';q,q') + \int \frac{d^2\mathbf{k}}{(2\pi)^2} K(p,p';k,p+p'-k)$$
$$\cdot \frac{i}{p_0+p_0'-\frac{\mathbf{k}^2}{2m}-\frac{(\mathbf{p}+\mathbf{p}'-\mathbf{k})^2}{2m}+i\epsilon}\Gamma(k,p+p'-k;q,q'). \tag{3.21}$$

We find it convenient to work in the center of mass frame where

$$\mathbf{p}' = -\mathbf{p}, \quad \mathbf{q}' = -\mathbf{q}, \quad p_0 + p_0' = q_0 + q_0' \equiv E. \tag{3.22}$$

Then the simplified notations $\Gamma \to \Gamma(\mathbf{p},\mathbf{q};E)$, $K \to K(\mathbf{p},\mathbf{q})$, etc. for the corresponding quantities should suffice, and Eq. (3.21) becomes

$$\Gamma(\mathbf{p},\mathbf{q};E) = K(\mathbf{p},\mathbf{q}) + \int \frac{d^2\mathbf{k}}{(2\pi)^2} K(\mathbf{p},\mathbf{k}) \frac{i}{E-\frac{\mathbf{k}^2}{m}+i\epsilon}\Gamma(\mathbf{k},\mathbf{q};E). \tag{3.23}$$

If we now decompose Γ and K as

$$\Gamma(\mathbf{p},\mathbf{q};E) = \sum_n \Gamma^n(|\mathbf{p}|,|\mathbf{q}|,E)e^{in\varphi}, \quad K(\mathbf{p},\mathbf{q}) = \sum_n K^n(|\mathbf{p}|,|\mathbf{q}|)e^{in\varphi} \tag{3.24}$$

(φ is the angle between the incoming and outgoing momenta) and insert these into Eq. (3.23), it follows after the angle integration that

$$\Gamma^n(|\mathbf{p}|,|\mathbf{q}|,E) = K^n(|\mathbf{p}|,|\mathbf{q}|) + \int \frac{d|\mathbf{k}|^2}{4\pi} K^n(|\mathbf{p}|,|\mathbf{k}|) \frac{i}{E-\frac{\mathbf{k}^2}{2m}+i\epsilon}\Gamma^n(|\mathbf{k}|,|\mathbf{q}|,E). \tag{3.25}$$

The n-th partial wave part of Γ being obtained by iterating K^n only, we are entitled to consider each partial wave contribution separately. As discussed in Ref. [16], non-s-wave (i.e., $n \neq 0$) parts can be shown to be finite order by order while the series for Γ^0 (obtained by iterating K^0) is not. This in turn implies that renormalization is necessary only for the s-wave amplitude Γ^0. So, for our purpose (i.e., to compare the field theoretic results with those of Sec. II), it should suffice from now on to confine our attention to the analysis of the s-wave amplitude, that is, to the $n = 0$ case with the intergral equation (3.25). Note that, from Eqs. (3.18a)-(3.18c), we have the s-wave contribution of the kernel given as

$$K^0 = K_a^0 + K_b^0 + K_c^0, \tag{3.26a}$$

$$K_a^0 = 0, \tag{3.26b}$$

$$K_b^0 = -i\frac{e^4}{4\pi m\kappa^2} \ln \frac{\Lambda^2}{L(|\mathbf{p}|,|\mathbf{q}|)^2}, \tag{3.26c}$$

$$K_c^0 = -i\lambda, \tag{3.26d}$$

where $L(|\mathbf{p}|,|\mathbf{q}|)$ denotes the larger of $|\mathbf{p}|$ and $|\mathbf{q}|$.

We will organize the series obtained by iterating the integral equation for Γ^0 in the following way. Let $\bar{\Gamma}^0$, Γ_{cross} and Γ_{bubble} denote the contributions given schematically by

$$\bar{\Gamma}^0 = K_b^0 + \int K_b^0(i\Delta)(i\Delta)K_b^0 + \int K_b^0(i\Delta)(i\Delta)K_b^0(i\Delta)(i\Delta)K_b^0 + \cdots, \tag{3.27}$$

$$\begin{aligned}\Gamma_{cross} = K_c^0 &+ \int K_c^0(i\Delta)(i\Delta)K_b^0 + \int K_b^0(i\Delta)(i\Delta)K_c^0 + \int K_c^0(i\Delta)(i\Delta)K_b^0(i\Delta)(i\Delta)K_b^0 \\ &+ \int K_b^0(i\Delta)(i\Delta)K_c^0(i\Delta)(i\Delta)K_b^0 + \int K_b^0(i\Delta)(i\Delta)K_b^0(i\Delta)(i\Delta)K_c^0 + \cdots,\end{aligned} \tag{3.28}$$

$$\begin{aligned}\Gamma_{bubble}^0 = &\int K_c^0(i\Delta)(i\Delta)K_c^0 + \int K_c^0(i\Delta)(i\Delta)K_b^0(i\Delta)(i\Delta)K_c^0 \\ &+ \int K_c^0(i\Delta)(i\Delta)K_b^0(i\Delta)(i\Delta)K_b^0(i\Delta)(i\Delta)K_c^0 + \cdots,\end{aligned} \tag{3.29}$$

respectively[4]. Denoting K_b^0 by the graph shown in Fig.6, these three amplitudes can be expressed graphically as in Fig.7. Then it is not difficult to see that the full s-wave amplitude Γ^0 is given by

$$\Gamma^0 = \bar{\Gamma}^0 + \Gamma_{contact} \tag{3.30}$$

with

$$\begin{aligned}\Gamma_{contact} &= \lambda\tilde{\Gamma}_{cross}\left\{1 + \lambda\tilde{\Gamma}_{bubble} + \left(\lambda\tilde{\Gamma}_{bubble}\right)^2 + \cdots\right\} \\ &= \frac{\lambda\tilde{\Gamma}_{cross}}{1-\lambda\tilde{\Gamma}_{bubble}},\end{aligned} \tag{3.31}$$

where we have defined $\tilde{\Gamma}_{cross} \equiv \frac{1}{\lambda}\Gamma_{cross}$ and $\tilde{\Gamma}_{bubble} \equiv \frac{i}{\lambda^2}\Gamma_{bubble}$. Note that all contact interaction terms are included in the quantity $\Gamma_{contact}$ (as defined by Eq. (3.31)). On the other hand, $\bar{\Gamma}^0$ has no dependence on the contact coupling λ.

Let us now look into the cutoff dependence of the quantities $\bar{\Gamma}^0, \tilde{\Gamma}_{cross}$ and $\tilde{\Gamma}_{bubble}$. Writing $x = \Lambda^2, \alpha = \frac{e^2}{2\pi\kappa}$ and using the relations (see Eqs. (3.26c) and (3.26d))

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[4]The quantity $\bar{\Gamma}^0$ given by Eq. (3.27) was denoted as Γ^0_{AB} in Ref. [16], for this amplitude was identified somewhat mistakenly with the s-wave part of the Aharonov-Bohm (AB) amplitude obtained under the hard-core boundary condition in that paper.

$$x\frac{d}{dx}K_b^0 = \frac{\pi\alpha^2}{m\lambda}K_c^0, \qquad x\frac{d}{dx}K_c^0 = 0, \tag{3.32}$$

we can readily deduce from the integral equations (3.27)-(3.29) the following relationships between these quantities:

$$x\frac{d\bar{\Gamma}^0}{dx} = \frac{\pi\alpha^2}{m}\tilde{\Gamma}_{cross}, \tag{3.33a}$$

$$x\frac{d\tilde{\Gamma}_{cross}}{dx} = \frac{2\pi\alpha^2}{m}\tilde{\Gamma}_{bubble}\tilde{\Gamma}_{cross}, \tag{3.33b}$$

$$x\frac{d\tilde{\Gamma}_{bubble}}{dx} = -\frac{m}{4\pi} + \frac{\pi\alpha^2}{m}(\tilde{\Gamma}_{bubble})^2. \tag{3.33c}$$

Note that the first term in the right hand side of (3.33c) originates from the cutoff dependence of the leading-order amplitude $\int K_c^0(i\Delta)(i\Delta)K_c^0$ from Γ_{bubble} (see (3.29)). Solving the differential equation (3.33c), we have

$$\tilde{\Gamma}_{bubble} = \left(\frac{m}{2\pi|\alpha|}\right)\frac{1-(d_1x)^{|\alpha|}}{1+(d_1x)^{|\alpha|}}, \tag{3.34}$$

where d_1 is a certain expression which is independent of x. Using Eq. (3.34) with Eq. (3.33b), we also find

$$\tilde{\Gamma}_{cross} = d_2\frac{(d_1x)^{|\alpha|}}{\{1+(d_1x)^{|\alpha|}\}^2} \tag{3.35}$$

and then, from Eq. (3.35) and Eq. (3.33a),

$$\bar{\Gamma}^0 = -\left(\frac{\pi|\alpha|d_2}{m}\right)\frac{1}{1+(d_1x)^{|\alpha|}} + d_3, \tag{3.36}$$

where we introduced two x-independent integration constants d_2 and d_3. Actually, by a simple dimensional reason, d_2 and d_3 may depend on α only while d_1 can be put as

$$d_1 = \left[\frac{e^{i\pi/2}}{p}f(\alpha)\right]^2 \tag{3.37}$$

(p is the magnitude of the *relative* momentum), with $f(\alpha) = 1 + \mathcal{O}(\alpha)$ on the basis of the result in lowest non-trivial order.

Inserting the expressions (3.34) and (3.35) into Eq. (3.31) yields the following expression for $\Gamma_{contact}$:

$$\Gamma_{contact} = \lambda d_2 \frac{\frac{(d_1 x)^{|\alpha|}}{1+(d_1 x)^{|\alpha|}}}{1 - \frac{\lambda m}{2\pi|\alpha|} + (1 + \frac{\lambda m}{2\pi|\alpha|})(d_1 x)^{|\alpha|}} \tag{3.38}$$

The desired full s-wave amplitude, given by Eq. (3.30), follows immediately from Eqs. (3.36) and (3.38). Since the s-wave two-particle scattering amplitude can be identified with $-2i\Gamma^0$, we now see that our field-theoretic analysis leads to

$$\begin{aligned} A_s(p) &= -2id_3(\alpha) - \frac{2\pi i|\alpha|}{m} d_2(\alpha) \frac{\lambda - \frac{2\pi|\alpha|}{m}}{\frac{2\pi|\alpha|}{m} - \lambda + (\frac{2\pi|\alpha|}{m} + \lambda)(d_1 x)^{|\alpha|}} \\ &= -2id_3(\alpha) - \frac{2\pi i|\alpha|}{m} d_2(\alpha) \frac{\lambda - \lambda_0}{\lambda_0 - \lambda + (\lambda_0 + \lambda)[\frac{\Lambda}{p} e^{i\pi/2} f(\alpha)]^{2|\alpha|}}, \end{aligned} \tag{3.39}$$

where, in the second expression, $\lambda_0 = \frac{2\pi|\alpha|}{m}$ and we have made use of the form (3.37). Based on this expression, we notice that if $\lambda = \pm\lambda_0 = \pm\frac{2\pi|\alpha|}{m}(= \pm\frac{e^2}{m\kappa})$, any dependence on the ultraviolet cutoff Λ disappears from $A_s(p)$ and the resulting amplitudes exhibit *scale invariance*. (just as was the case with the quantum mechanical expression for those λ-values) For these special values of λ, the scattering amplitude is given by a finite perturbation theory and the system requires no renormalization. This happens because divergences appearing in the perturbation theory of $\bar{\Gamma}^0$ get cancelled order by order by divergences resulting from contributions involving the contact interaction [16]. Now, at least for these finite-theory cases, we may be allowed to invoke the usual one-to-one correspondence existing between a nonrelativistic quantum field theory and the quantum mechanical approach [1] —viz., for $\lambda = \pm\lambda_0$, our amplitude (3.39) should match the result in Eq. (2.30). [We have not been able to verify this assertion directly, however.] Taking this for granted, we can now fix $d_3(\alpha)$ and $d_2(\alpha)$ in the above expression by using the result in (2.30) as

$$-2id_3(\alpha) = e^{-i\pi|\alpha|} - 1, \tag{3.40a}$$

$$-2id_3(\alpha) + \frac{2\pi i|\alpha|}{m} d_2(\alpha) = e^{i\pi|\alpha|} - 1, \tag{3.40b}$$

and this leads to the following expression for $A_s(p)$:

$$A_s(p) = e^{-i\pi|\alpha|} - 1 - (e^{i\pi|\alpha|} - e^{-i\pi|\alpha|}) \frac{\lambda - \lambda_0}{\lambda_0 - \lambda + (\lambda_0 + \lambda)[\frac{\Lambda}{p} e^{i\pi/2} f(\alpha)]^{2|\alpha|}} \tag{3.41}$$

For $\lambda \neq \pm\lambda_0$, we must renormalize the theory to obtain the scattering amplitude which has no explicit dependence on the cutoff. That is, we will regard λ as a bare coupling and elect to introduce λ_{ren}, the renormalized coupling, by the relation

$$\frac{\lambda_0 + \lambda}{\lambda_0 - \lambda}(\Lambda f(\alpha))^{2|\alpha|} = \frac{\lambda_0 + \lambda_{ren}}{\lambda_0 - \lambda_{ren}}\mu^{2|\alpha|}. \tag{3.42}$$

Here, μ is the normalization scale. Then the amplitude (3.41) can be recast into the form

$$A_s(p) = e^{-i\pi|\alpha|} - 1 - (e^{i\pi|\alpha|} - e^{-i\pi|\alpha|})\frac{\lambda_{ren} - \lambda_0}{\lambda_0 - \lambda_{ren} + (\lambda_0 + \lambda_{ren})(\mu/p)^{2|\alpha|}e^{i\pi|\alpha|}} \tag{3.43}$$

This is in complete agreement with the quantum mechanical expression (2.30), only if our field theoretic renormalized coupling λ_{ren} is taken to be related to the self-adjoint extension parameter θ by Eq. (2.29). We may now assert that the quantum field theory defined through the action (3.1) and renormalized as above provides an equivalent description of many anyon quantum mechanics with a general boundary condition as considered in the previous section.

Before closing this section, we shall emphasize once again the role of the contact interaction terms in the quantum description of anyons. As was explained in Sec. II, we need them to implement dynamically (i.e., through the Hamiltonian) a suitable boundary condition at the two-anyon coincidence point. As such, their presence is in no way an artifact of perturbation theory. They essentially go over to the field theoretic description, where one usually does not consider the boundary condition separately. In fact, without including the contact term in the Lagrangian density, the given field theory is not renormalizable and hence does not lead to a well-defined theory. The equivalence between the first- and second-quantized approaches can be established only when we include the appropriate contact interaction term. For instance, in the special case of anyons satisfying the hard-core boundary condition, the Lagrangian density of the corresponding field theory reads

$$\mathcal{L} = \frac{\kappa}{2}\partial_t \mathbf{A} \times \mathbf{A} - \kappa A_0 B + \phi^\dagger(iD_t + \frac{\mathbf{D}^2}{2m})\phi - \frac{e^2}{2m\kappa}\phi^\dagger\phi^\dagger\phi\phi, \tag{3.44}$$

and this happens to be an ultraviolet finite theory.

IV. SUMMARY AND DISCUSSIONS

Due to the singular nature of anyon interaction at short distance, one has a well-defined many anyon quantum mechanics only after a suitable boundary condition at the two-anyon coincidence point has been chosen. This introduces a new free parameter—the self-adjoint extension parameter θ in the theory, which serves to specify the chosen boundary condition. We have shown that this boundary condition, in its full generality, can be implemented dynamically by introducing a suitable contact interaction term in the anyon Hamiltonian. This system admits a quantum field theoretic description in the form of a Chern-Simons gauge theory, and we have here shown that the $\phi^\dagger\phi^\dagger\phi\phi$-type contact interaction assumes a crucial role not only in securing a well-defined theory but also in realizing the full equivalence with the quantum mechanical approach. The strength of the renormalized contact coupling in the field-theoretic description is related to the self-adjoint extension parameter which encodes the quantum mechanical boundary condition.

We have a few comments to make. First of all, note that more general kinds of anyons (other than the ones we discussed here) are possible, such as those obeying so-called matrix(or mutual) statistics [21] and also those obeying non-Abelian statistics [22]. Both quantum-mechanical and field-theoretic descriptions for these generalized (but still nonrelativistic) anyons were discussed by various authors [23,24] without paying due attention to the contact interaction terms. These must be corrected along the line discussed in this paper. [In this regards, see especially Ref. [25] where the related issue is studied in non-Abelian Chern-Simons field theory with the help of some lower-order perturbative calculations.] Another problem deserving more study is to look at related issues from the viewpoint of *relativistic* Chern-Simons field theory, as was considered recently in Ref. [26] whithin one-loop approximation. Finally, we need to have more information on those specific features of an anyon system which depend crucially on the self-adjoint extension parameter(or, equivalently, on the coupling strength of the $\phi^\dagger\phi^\dagger\phi\phi$ interaction in the field theoretic approach). After all, if anyon play a role in real physical phenomena, it will be an experimental question to

determine what specific boundary condition the given anyons satisfy.

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FIGURECAPTIONS

FIG. 1. Basic vertices of the theory defined by the action (3.1)

FIG. 2. Vanishing diagrams

FIG. 3. Graphical representation of the effective two-particle interaction Γ

FIG. 4. Representation of the full kernel K in terms of three different contributions

FIG. 5. Diagrams yielding vanishing contribution to the two-particle irreducible kernel

FIG. 6. Graphical representation of K_b^0

FIG. 7. Graphical representation of $\bar{\Gamma}^0$, Γ_{cross} and Γ_{bubble}

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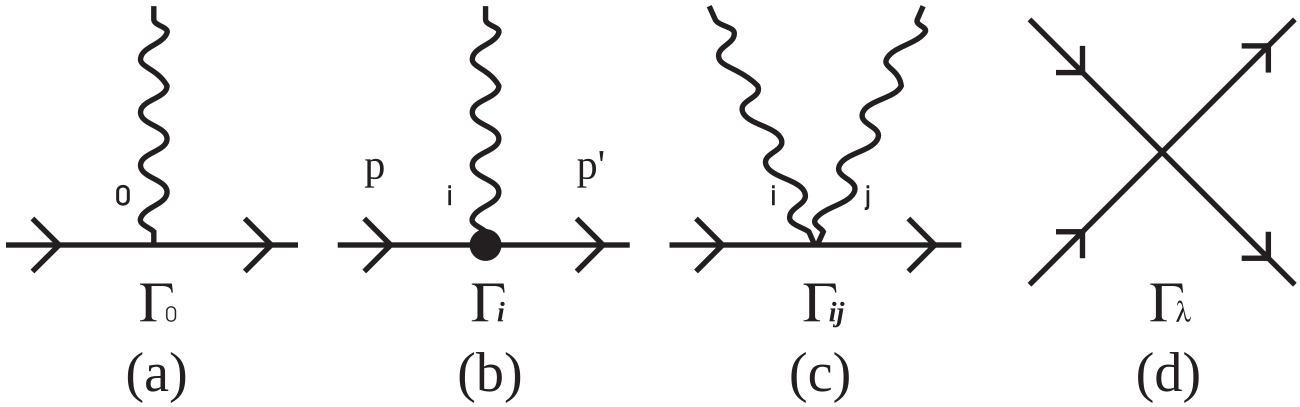

FIGURE 1

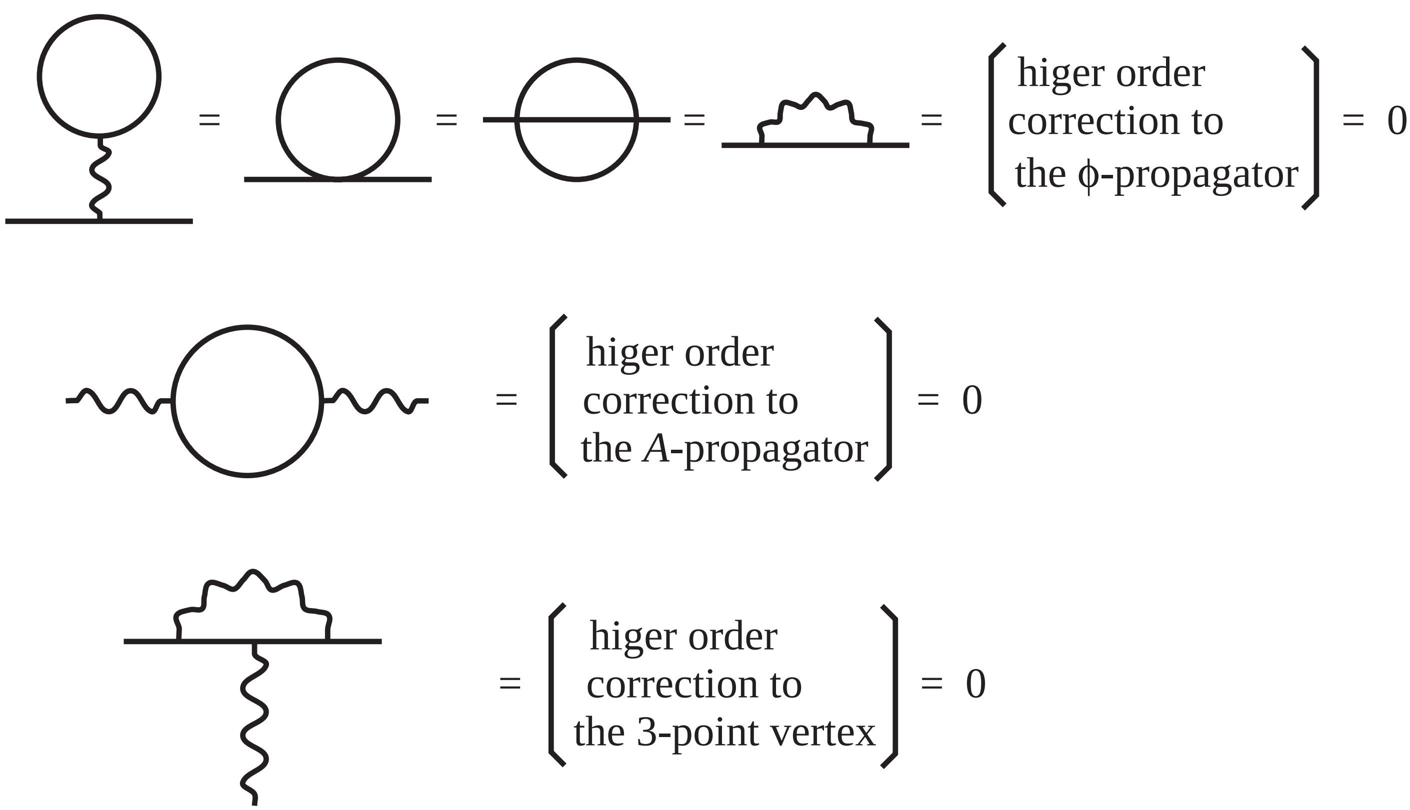

FIGURE 2

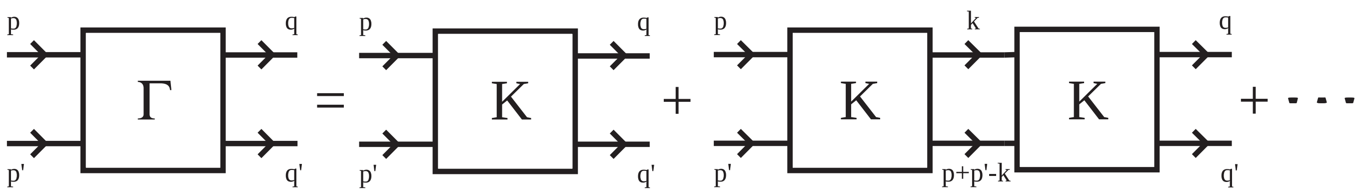

FIGURE 3

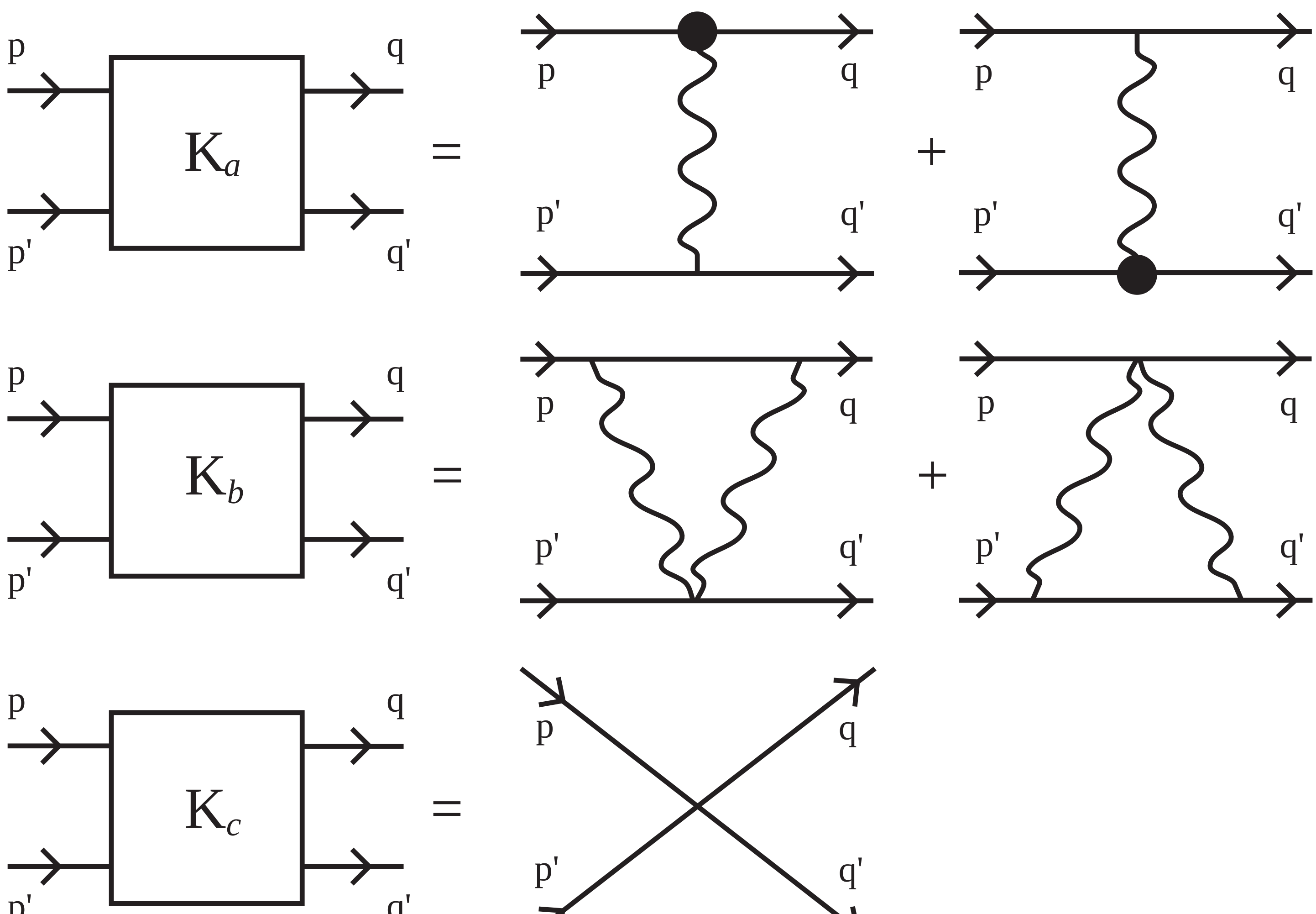

FIGURE 4

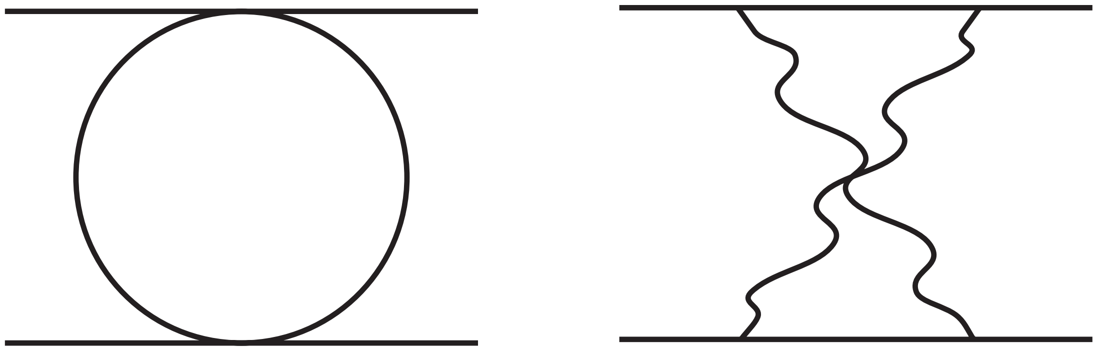

FIGURE 5

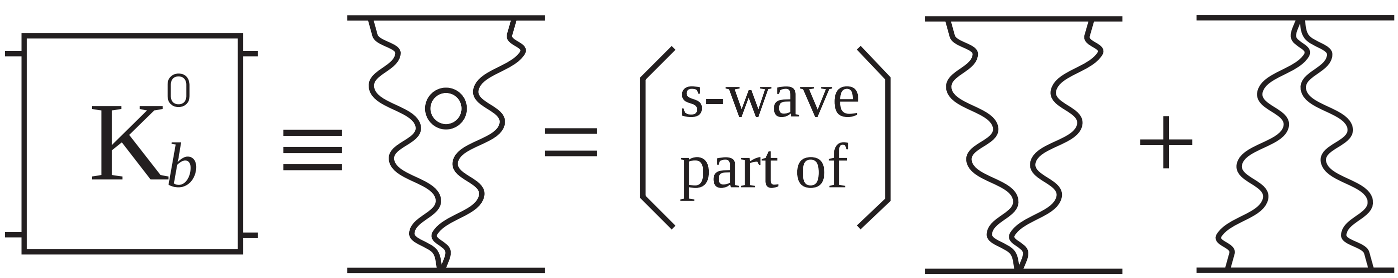

FIGURE 6

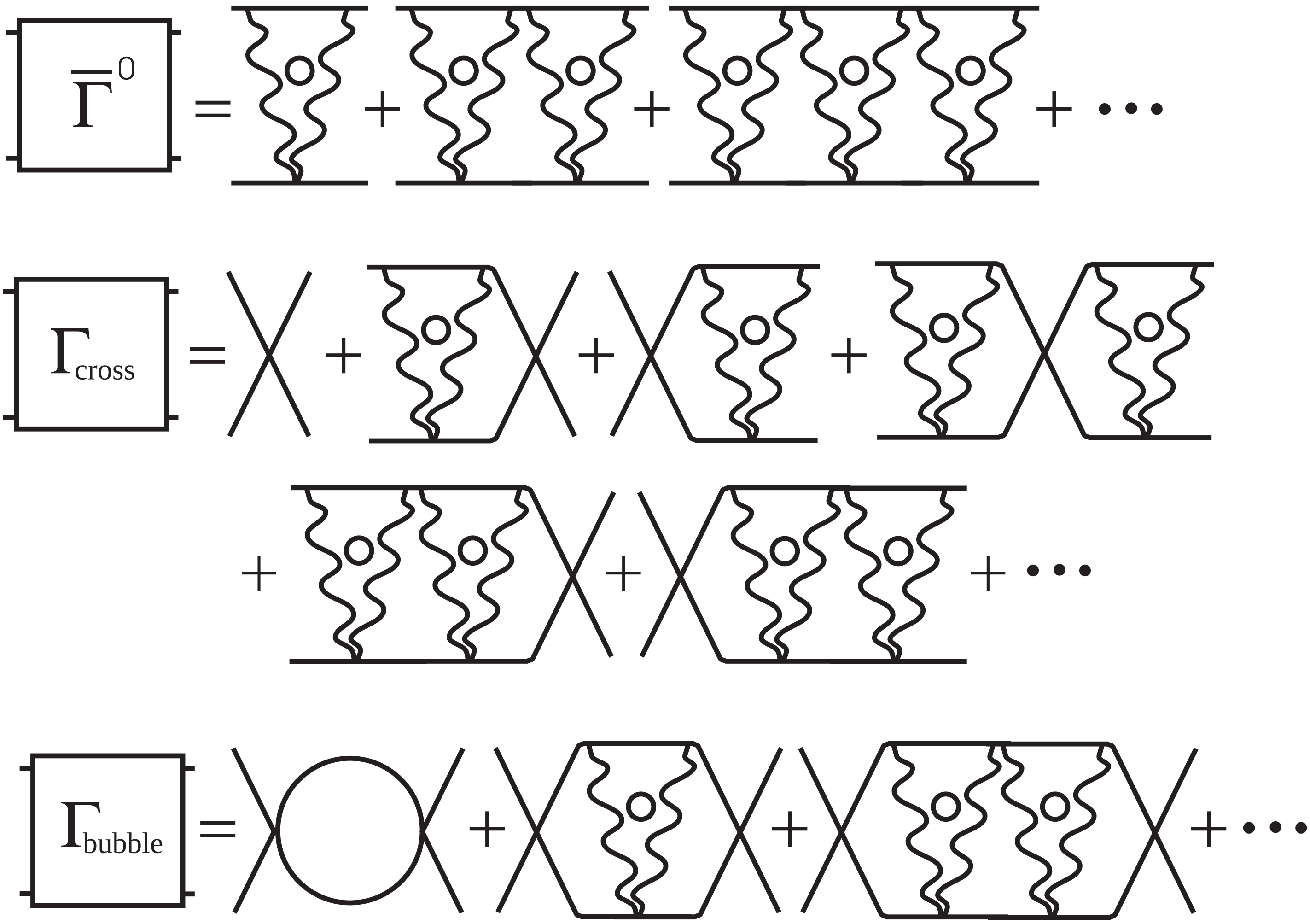

FIGURE 7